Lecture Notes

B. Sc. Mathematics (H)

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Part-2

Paper-III

B. Sc. Maths (H)

EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

Euler's Theorem on Homogeneous Functions

Homogeneous Functions:

A function of the form

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots\dots\dots + a_{n-1}xy^{n-1} + a_ny^n$$

in which each term is of degree n is said to be a homogeneous function of degree n in x and y.

A function f(x, y) is said to be homogeneous of degree 'n' in x and y, if it can be written in any one of the following forms:

$$f(\lambda x, \lambda y) = \lambda^n f(x, y),\ f(x, y) = x^n g\left(\frac{y}{x}\right),\ f(x, y) = y^n \varphi\left(\frac{x}{y}\right)$$

A function of the form $x^m y^n g(x/y)$ or $x^m y^n g(y/x)$ is also homogeneous of degree (m+n). Here degree of g is considered as zero. The degree of a homogeneous function may be a rational number.

A function f(x, y, z) of three variables is said to be homogeneous of degree n, if it can be written as

$$f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x, y, z),\ f(x, y, z) = x^n g\left(\frac{y}{x}, \frac{z}{x}\right), \text{ etc.}$$

Example: $\sin(x^2y + xy^2)$ is not homogeneous.

$x^4 \tan\left(\frac{y}{x}\right)$ is homogeneous of degree 4.

$\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}$ is homogeneous of degree $\frac{1}{3}-\frac{1}{2}=\frac{-1}{6}$.

Euler's Theorem on Homogeneous Function:

If z = f(x, y) is a homogeneous function in x and y of degree n, then

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz.$$

Proof: We can write $z = x^n f(y/x)$ or $z = x^n f(t)$, where t = y/x.

Then

$$\frac{\partial z}{\partial x} = nx^{n-1}f(t) + x^n f'^{(t)}\frac{\partial t}{\partial x} = nx^{n-1}f(t) + x^n f'^{(t)}\left(\frac{-y}{x^2}\right)$$

$$= nx^{n-1}f(t) - x^{n-2}f'^{(t)}y$$

and

$$\frac{\partial z}{\partial y} = x^n f'^{(t)}\frac{\partial t}{\partial y} = x^n f'^{(t)}\left(\frac{1}{x}\right)$$

$$= x^{n-1}f'^{(t)}$$

Adding them we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nxx^{n-1}f(t) - xx^{n-2}f'^{(t)}y + yx^{n-1}f'^{(t)} = nz$$

Extension for Three Variable Homogeneous Function:

If u = f(x, y, z) is a homogeneous function in x, y, and z of degree n, then we can write it as $u = x^n f(y/x, z/x)$ and then we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu\ .$$

Extension for n Variable Homogeneous Function:

If u = f(x_1, x_2, x_3, ……. , x_i, ….. , x_n) is a homogeneous function of degree n, then

$$x_1\frac{\partial u}{\partial x_1}+x_2\frac{\partial u}{\partial x_2}+x_3\frac{\partial u}{\partial x_3}+........+x_i\frac{\partial u}{\partial x_i}+........+x_n\frac{\partial u}{\partial x_n}=nu\ .$$

Deductions from Euler's Theorem on Homogeneous Function for Second Derivatives:

If u = f(x, y) is a homogeneous function in x and y of degree n, then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$$

i.e.,

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 u=n(n-1)u$$

Proof: We know that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \qquad (1)$$

Differentiating (1) partially w.r.to a and y separately, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x} \qquad (2)$$

and

$$y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}+x\frac{\partial^2 u}{\partial y\partial x}=n\frac{\partial u}{\partial y} \qquad (3)$$

Multiplying (2) by x and (3) by y and then adding them, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}+xy\frac{\partial^2 u}{\partial y\partial x}=nx\frac{\partial u}{\partial x}+ny\frac{\partial u}{\partial y}$$

Now using (1) and the fact that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

Deductions from Euler's Theorem:

If z is a homogeneous function in x and y of degree n and z is a function of u as

z = φ(x, y) and z = f(u)

then we have

(i) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)}$

(ii) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$,

where, $g(u) = n \frac{f(u)}{f'(u)}$.

Proof: Do Yourself

Note: If z is a function of x and y and its degree cannot be determined, then we express it as z=φ(x, y) and z=f(u). Thereafter we apply the above theorem.

Extension for Function of Three Variables: If v is a homogeneous function in x, y, and z of degree n and v = f(u). Then we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=n\frac{f(u)}{f'(u)}$$

Examples:

1. If we have given that $z = x^4 log\frac{\sqrt[3]{y}-\sqrt[3]{x}}{\sqrt[3]{y}+\sqrt[3]{x}}$, then find the value of $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$.

2. If given that $u = x^4y^2\sin^{-1}\frac{x}{y} + \log x - \log y$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 6x^4y^2\sin^{-1}\frac{x}{y}$.

3. If we have $z = \log\left(\frac{x^2+y^2}{x+y}\right)$, then find the values of

(a) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$ and (b) $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2}$.