Lecture Notes

B. Sc. Maths (H)

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Part-1

Paper-I

B. Sc. Maths (H)

GENERAL EQUATION OF PARABOLAS

General Equation of a Parabola

Let us consider the axis parallel to x-axis. Let us take the transformations

$$X = x - \alpha, \quad Y = y - \beta$$

Then the equation of the parabola in new coordinate system becomes $Y^2 = 4aX$, which is the standard equation in X, Y co-ordinates.

Therefore in X, Y co-ordinates, we have:

$$(y - \beta)^2 = 4a\,(x - \alpha)$$

For which we have

Vertex = (0, 0)

Focus = (a, 0)

Equation of the axis Y = 0

Equation of the directrix X + a = 0.

Therefore in x, y co-ordinates we have using $x = X + \alpha$ and $y = Y + \beta$:

Vertex = (α, β)

Focus = $(a + \alpha, \beta)$

Equation of the axis $y - \beta = 0$

Equation of the directrix $x - \alpha + a = 0$.

Similarly the general equation of a parabola whose axis is parallel to y-axis is

$$(x - \alpha)^2 = 4a\,(y - \beta)$$

Position of a Point in Relation to a Parabola:

Let the equation of the parabola be $y^2 = 4ax$ and the point considered is (x_1, y_1). Then the point lies outside, upon or inside the parabola according as: $y_1^2 - 4ax_1$ is greater than, equal to or less than zero.

Examples:

1. Find the equation to the parabola whose focus is (5, 3) and directrix, the line $3x - 4y + 1 = 0$.

[Hint: Use $(x-5)^2 + (y-3)^2 = \left[\pm \frac{3x-4y+1}{\sqrt{3^2+(-4)^2}}\right]^2$ to get the required equation $16x^2 + 9y^2 + 24xy - 256x - 142y + 849 = 0$]

2. Find the equation of the parabola having the vertex at (0, 1) and the focus at (0, 0).

[Hint: Find the equation of the directrix $y - 2 = 0$ and then apply the focus-directrix property]

3. Prove that the equation of the parabola whose focus is (0, 0) and tangent at the vertex is $x - y + 1 = 0$ is $x^2 + y^2 + 2xy - 4x + 4y - 4 = 0$.

4. Find the vertex, focus, axis, and latus rectum of the parabola $4y^2 + 12x - 20y + 67 = 0$.

5. Find the vertex and directrix of the parabola $y^2 - 3x - 2y + 7 = 0$.

Part-2

Paper-III

B. Sc. Maths (H)

TAYLOR'S SERIES EXPANSION

Taylor's Theorem:

Let f(x) be a function of x and h be small. If the function f(x+h) can be expanded as an infinite convergent series of terms of positive integral powers of h, then this expansion is given by

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + ... + \frac{h^n}{n!} f^n(x) + ...$$

where $f^n(x)$ stands for the nth derivative of f(x+h) with respect to (x+h), when (x+h) is replaced by x. The series on R.H.S. is known as Taylor's series.

Note: Differentiation of f(x+h) with respect to (x+h) or h gives the same result, i.e.

$$\frac{d}{d(x+h)} f(x+h) = f'(x+h) \text{ and } \frac{d}{dh} f(x+h) = f'(x+h).$$

Expansion in powers of (x-a) by Taylor's theorem:

Let a function f(x) have derivatives of all orders in $\alpha < x < \beta$. Then for each positive integer n and each a in the interval, the function f(x) may be expanded in powers of (x-a) by Taylor's theorem as [by putting h=(x-a)].

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + ... + \frac{(x-a)^n}{n!} f^n(a) + R_{n+1}(x),$$

where $f^n(a)=\dfrac{d^n f}{dx^n}$ and the remainder $R_{n+1}(x)$ is given by $R_{n+1}(x)=\dfrac{(x-a)^{n+1}}{(n+1)!}f^{n+1}(\xi)$ for some ξ between a and x.

Note:

(1) Taylor's theorem becomes the Taylor's series for f(x) when n is allowed to become infinite.

(2) If the remainder term is neglected in Taylor's theorem, the result is called the Taylor's polynomial approximation to f(x) of degree n.

(3) Taylor's theorem reduces to Maclaurin's theorem if a=0 and if we allow n to become infinite in Maclaurin's theorem, it becomes the Maclaurin's series for f(x).

(4) When Taylor's theorem is terminated with the term $R_1(x)$, corresponding to n=0,

$$\frac{f(x)-f(a)}{(x-a)}=f'(\xi),\ a\le\xi\le x,$$

called the mean value theorem for derivatives.

Corollary: Putting x=0 and h=x in Taylor's series, we obtain Maclaurin's series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+..........+\frac{x^n}{n!}f^n(0)+.....$$

Thus Maclaurin's series can be obtained as a particular case of Taylor's series.

Conditions under which the Taylor's Theorem is Valid:

(1) The function f(x) and its derivatives must be finite and continuous in the range of definition of f(x).

(2) The series on the R.H.S. must be convergent for which the remainder term $R_n \to 0$ as $n \to \infty$,

where, $R_n = \frac{h^n}{n!} f^n(x+\theta h)$, $0<\theta<1$.

Note: Taylor's series can be derived from Maclaurin's series. Taylor's and Maclaurin's series are not essentially different.

Examples:

1. Expand log{sin(x+b)} in ascending powers of b as far as the term involving b^3.

2. Prove by Taylor's theorem that

$$tan^{-1}(x+b) = tan^{-1}x + bsin\alpha \frac{sin\alpha}{1} - (bsin\alpha)^2 \frac{sin2\alpha}{2} + (bsin\alpha)^3 \frac{sin3\alpha}{3} - \cdots$$

where $x = cot\alpha$.