**Lecture Notes**
**02/04/2020**

**Dr. Dharmendra Kumar Yadav**
Assistant Professor of Mathematics
M. L. S. College, Sarisab-Pahi, Madhubani
R. K. College, Madhubani



## Part-1

Paper-I
B. Sc. Maths (H)

# EQUATION OF TANGENT ON CONIC SECTION

## Part-2

Paper-III
B. Sc. Maths (H)

# CONVERGENCY AND DIVERGENCY OF INFINITE SERIES

**Part-2**
**Paper-III**
**B. Sc. Maths (H)**

## CONVERGENCY AND DIVERGENCY OF INFINITE SERIES

**Convergence of Geometric Series**: The geometric series

$$\sum ar^{n-1} = a + ar + ar^2 + ar^3 + \text{..........}$$

is (1) Convergent if $|r| < 1$, i.e. $-1 < r < 1$,

(2) Divergent if $r \geq 1$, and

(3) Oscillatory if $r \leq -1$. Oscillates finitely if $r = -1$ and Oscillates infinitely if $r < -1$.

**Theorem (Necessary Condition for Convergence of a Series)**: If $\sum a_n$ is a convergent series of positive terms, then $\lim_{n \to \infty} a_n = 0$.

But its converse is not true.

**For Example**: $\sum \frac{1}{n}$ is divergent.

**Note**: This theorem is a necessary condition and not sufficient for the convergence of $\sum a_n$, i.e. the converse of the theorem is not true. In other words, even if $\lim_{n \to \infty} a_n = 0$ for a for a given series, the series $\sum a_n$ may not be convergent.

**Example**: For the series $\sum \frac{1}{n}$, $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0$, but the given series is divergent.

**Working Rule to Test for Divergent Series**: The given series $\sum a_n$ is divergent, if

$$\lim_{n \to \infty} a_n \neq 0,$$

i.e. if $\lim_{n \to \infty} a_n \neq 0$ for the given series, then the positive term series $\sum a_n$ must be divergent.

**Example**: $\sum n$

**Convergence or Divergence of p-series/ Auxiliary Series/Harmonic Series**: The series

$$\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots + \frac{1}{n^p} + \ldots\ldots$$

is (1) Convergent if p>1, and

(2) Divergent if $p \leq 1$.

The infinite series $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots + \frac{1}{n^p} + \ldots\ldots$ is known as Harmonic series or p-series or auxiliary series or Hyper Harmonic series.

**Higher Auxiliary Series**: The series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ is convergent if p>1 and divergent if p≤1.

**Example**: Discuss the nature of the series $\sum_{n=2}^{\infty} \frac{1}{(n \log n)^p}$.

**Tests for the Convergence & Divergence of Positive Term Series**: We have so far discussed the convergence or divergence of a series by evaluating the sum of first n terms of the series. However, it is not easy to calculate the sum of first n terms of every series. Thus it is not possible to determine the nature of every series by direct application of definition. Hence various other methods have been derived to determine the convergence or divergence of a series.

**Comparison Test ($V_n$-method) (First Form)**: Let the positive term series under consideration is $\sum U_n$ and let $\sum V_n$ be another positive term series (particular form of p-series) whose nature is already known by p-series, such that

$$\lim_{n \to \infty} \frac{U_n}{V_n} = k, (k \neq 0)$$

Then, both of the series will be either convergent or divergent simultaneously according as the series $\sum V_n$ .

**Comparison Test (Second Form)**: Let the positive term series under consideration is $\sum U_n$ and let $\sum V_n$ be another positive term series (particular form of p-series) whose nature is already known by p-series, then the series

(1) $\sum U_n$ is convergent if $\sum V_n$ is convergent and $\lim_{n \to \infty} \frac{U_n}{V_n} = 0$, and

(2) $\sum U_n$ is divergent if $\sum V_n$ is dinvergent and $\lim_{n \to \infty} \frac{U_n}{V_n} = \infty$.

**Comparison Test (Third Form)**: Let the positive term series under consideration is $\sum U_n$ and let $\sum V_n$ be another positive term series (particular form of p-series) whose nature is already known by p-series, then the series $\sum U_n$ is

(1) Convergent if the series $\sum V_n$ is convergent and $U_n \le V_n, \forall n \in N$, and

(2) Divergent if the series $\sum V_n$ is divergent and $\mathrm{U_n \ge V_n, \forall n \in N}$.

**Comparison Test (Fourth Form)**: Let the positive term series under consideration is $\sum U_n$ and let $\sum V_n$ be another positive term series (particular form of p-series) whose nature is already known by p-series, then

(1) The series $\sum U_n$ is convergent, if the series $\sum V_n$ is convergent and $\frac{U_n}{U_{n+1}} > \frac{V_n}{V_{n+1}}$, and

(2) The series $\sum U_n$ is divergent, if the series $\sum V_n$ is divergent and $\frac{U_n}{U_{n+1}} < \frac{V_n}{V_{n+1}}$.

**Working Rule to Apply Comparison Test**: To apply the comparison test, we need to have some series $\sum V_n$ whose behavior is already known. This series is always a particular form of p-series. This test is very useful when degree of n in $U_n$ can be determined.

**How to find $\sum V_n$ :**

(1) When $U_n$ is in the form of a fraction of two polynomials, to select auxiliary series $\sum V_n = \sum \frac{1}{n^p}$, it should be noted that p=difference in degree of n in denominator and

numerator of $U_n$, i.e. to find $V_n$, let us take $U_n$, the nth term of the given series and retain only the highest powers of n in both numerator and denominator. Let us denote this result by $V_n$ and then write $V_n$ as $V_n = \frac{1}{n^p}$, i.e.

$$V_n = \frac{n^{highest\ power}\ in\ numerator\ of\ U_n}{n^{highest\ power}\ in\ denominator\ of\ U_n}.$$

**(2)** When $U_n$ can be expanded in ascending powers of $\frac{1}{n}$, then to get $V_n$, we should write only the lowest power of $\frac{1}{n}$.

**Example**: (1) $U_n = \sqrt{n^4+1} - \sqrt{n^4-1} = n^2\left\{\left(1+\frac{1}{n^4}\right)^{1/2} - \left(1-\frac{1}{n^4}\right)^{1/2}\right\}$

$= \frac{1}{n^2} + highest\ power\ of\ \frac{1}{n}$

$\therefore Vn = \frac{1}{n^2}$

In this case, we can put $U_n$ in fraction form by multiplying numerator and denominator by the conjugate of $\sqrt{n^4+1} - \sqrt{n^4-1}$, i.e. by $\sqrt{n^4+1} + \sqrt{n^4-1}$. Then $U_n$ can be written as $U_n = \frac{2}{\sqrt{n^4+1} + \sqrt{n^4-1}}$. In this case $V_n = \frac{1}{n^2}$

(2) $U_n = \left(n^3+1\right)^{1/3} - n$. For this we have $V_n = \frac{1}{n^2}$. Expand $U_n$ in ascending powers of (1/n) and then divide by $V_n$ and take the limit, we get the limit (1/3).

**Part-1**
**Paper-I**
**B. Sc. Maths (H)**

## EQUATION OF TANGENT ON CONIC SECTION

### Rule to remember the Equation of a Tangent

We know that the equation of the tangent to

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

at $(x_1, y_1)$ is

$$axx_1 + b(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

which can be written conveniently by making the following changes in general equation of second degree:

(1) Replace one x by $x_1$ in $x^2$ (= x.x) as $x.x_1$ and in 2x (= x + x) as $x + x_1$.

(2) Replace one y by $y_1$ in $y^2$ (= y.y) as $y.y_1$ and in 2y (= y + y) as $y + y_1$.

(3) Replace 2xy (by xy + xy) by $xy_1 + x_1y$.

**Example:** The equation of the tangent to the conic

$$3x^2 + 4xy - 6y^2 - 3x + 8y + 2 = 0$$

at $(x_1, y_1)$ is

$$3xx_1 + 2\,(x_1y + xy_1) – 6yy_1 – (3/2)(x + x_1) + 4(y + y_1) + 2 = 0$$

### Particular Cases of Equation of Tangent for Conic Sections:

### Tangent to Parabola:

The equation of tangent to the parabola $y^2 = 4ax$ at the point $(x_1, y_1)$ is given by

$$yy_1 = 2a(x + x_1)$$

**Tangent to Ellipse:**

The equation of tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(x_1, y_1)$ is given by

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

**Tangent to Hyperbola:**

The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(x_1, y_1)$ is given by

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

**Proofs of all above three must be done.**

**Condition of Tangency:**

The condition that the line lx+my+n=0 may touch the conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Is

$$(blm - hnm - flm + gm^2)^2 = (am^2 - 2hlm + bl^2).(bn^2 - 2fmn + cm^2)$$

**Proof is needed for exam purpose.**

**Particular Cases:**

1. The condition that the line y = mx + c may touch the parabola $y^2 = 4ax$ is c = a/m.

**Note:** The equation of a tangent to the parabola $y^2 = 4ax$ is y = mx + (a/m) for every value of m.

2. The condition that the line y = mx + c may touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$c = \pm\sqrt{a^2m^2 + b^2}.$$

**Note:** The equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$y = mx \pm\sqrt{a^2m^2 + b^2}$$

for every value of m.

3. The condition that the line y mx + c may touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$c = \pm\sqrt{a^2m^2 - b^2}$$

**Note:** The equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$y = mx \pm\sqrt{a^2m^2 - b^2}$$

for every value of m.

**Example:** Find the equation of the tangent at the point (-2, 1) to the conic

$$x^2 + 2xy - y^2 + 2x + 4y + 1 = 0$$