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31/03/2020 **[MLS College]**

03/04/2020 **[R K College]**

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Part-1

Paper-II

Intersection of a Line and a Conic

Every line cuts a conic section in two points.

Let $P(x_1, y_1)$ be a given point through which a line passes. The equation to the line is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = r \qquad (1)$$

where l and m are constants such that $\frac{m}{l} = tan\theta$ = slope of the line and r is the algebraical distance of P from any point (x, y) on the line.

Then from (1)

$$x = x_1 + lr,\ y = y_1 + mr.$$

The line (1) meets the conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad (2)$$

The points of intersection of (1) and (2) are given by

$$a(x_1 + lr)^2 + 2h(x_1 + lr)(y_1 + mr) + b(y_1 + mr)^2 + 2g(x_1 + lr) + 2f(y_1 + mr) + c = 0$$

which implies

$$r^2(al^2 + 2hlm + bm^2) + 2r\{(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m\} + S_1 = 0 \qquad (3)$$

where $S_1 \equiv ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0$

Now the equation (3) is of second degree in r and accordingly gives two values of r.

These two values of r may be real and different, real and coincident or imaginary depending on the location of P with respect to the conic and the direction of the line.

Hence every straight line cuts a conic in two points.

Notes: (1) If r_1 and r_2 are the roots of the equation (3), then the points of intersection are given by $(x_1 + lr_1, y_1 + mr_1)$.

(2) A line is a tangent to a conic if it cuts the conic at coincident points. Therefore, if the roots of equation (3) are both equal to zero, then the line (1) becomes the tangent to (2) at the point (x_1, y_1).

Equation of Tangent: **[LNMU 2019 (H)]**

To find the equation of a tangent to the conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad (1)$$

at the point (x_1, y_1).

We know that the value of m of the tangent is

$$= \frac{dy}{dx} = -\frac{\partial S/\partial x}{\partial S/\partial y} \qquad (3)$$

Therefore the equation of the tangent at (x_1, y_1) is

$$y - y_1 = m\,(x - x_1)$$

$$y - y_1 = -\frac{\partial S/\partial x}{\partial S/\partial y}\,(x - x_1)$$

which imples that

$$(x - x_1)\left(\partial S/\partial x\right) + (y - y_1)\left(\partial S/\partial y\right) = 0 \qquad (4)$$

at (x_1, y_1).

But by (1) we have

$$\frac{\partial S}{\partial x} = 2ax + 2hy + 2g = 2(ax + hy + g)$$

Therefore at (x_1, y_1)

$$\frac{\partial S}{\partial x}_{(x_1, y_1)} = 2(ax_1 + hy_1 + g)$$

Similarly

$$\frac{\partial S}{\partial y}_{(x_1, y_1)} = 2(hx_1 + by_1 + f)$$

Therefore the equation (4) becomes

$$(x - x_1)(ax_1 + hy_1 + g) + (y - y_1)(hx_1 + by_1 + f) = 0$$

which implies that

$$x(ax_1 + hy_1 + g) + y\,(hx_1 + by_1 + f) + (gx_1 + fy_1 + c) = ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c \quad (5)$$

But the point (x_1, y_1) is on the conic S = 0. Therefore

$$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0 \qquad (6)$$

by (5) and (6), the equation to the tangent at (x1, y1) on the conic is

$$x(ax_1 + hy_1 + g) + y\,(hx_1 + by_1 + f) + (gx_1 + fy_1 + c) = 0$$

which can also be arranged as

$$axx_1 + byy_1 + h\,(xy_1 + x_1y) + g\,(x + x_1) + f\,(y + y_1) + c = 0$$

which is the required equation of the tangent to the conic.

Part-2

Paper-III

Lagrange’s Method of Undetermined Multipliers

Conditional Extremum: To find the maxima and minima of a function subject to one or two conditions or constraints. The general form of such type of examples is as follows:

Maximize or Minimize u=f(x, y) subject to g(x, y)=0.

Working Rule: In order to find the extremum value of u=f(x, y) under the given condition, one of the variables may be eliminated.

If we can solve the latter equation g(x, y)=o for y in terms of x and substitute it in the function f(x, y), then the problem reduces to that of finding the maxima or minima of a function of a single variable x.

However in complicated cases, it is not convenient to achieve the elimination of the one of the variables and usually Lagrange’s method of undetermined multipliers is used.

To Find the Maxima or Minima of a Function of Two Variables:

Let us consider the function u=f(x, y) subject to the condition g(x, y)=0.

Working Rule:

1. Given that u=f(x, y) …….. (1) and g(x, y)=0 ……. (2)

2. Let us form a Lagrange’s function F(x, y) = f(x, y) + λ g(x, y), where λ is a non-zero parameter to be determined, known as Lagrange’s multiplier or undetermined multiplier.

3. For stationary points of F we have dF=0. Therefore the stationary points of F are given by

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x}=0 \text{ and } \frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y}=0 \quad \ldots\ldots\ldots (3)$$

4. Eliminating x, y and λ from (2) and (3), we get an equation in u, the roots of which give the stationary values of u=f(x, y).

Generally we determine the values of x and y in terms of λ from equation (3) and put them in the given condition (2), which gives the values of λ. Again putting the values of λ in x and y obtained from (3) gives the stationary points. These points when substituted in the given function give the extremum values.

To Find the Maxima or Minima of a Function of Three Variables:

Let us consider the function u=f(x, y, z) subject to the conditions g(x, y, z)=0 and h(x, y, z).

Working Rule:

1. Given that u=f(x, y, z) …….. (1)

g(x, y, z)=0 ……. (2) and h(x, y, z)=0 ……. (3)

2. Let us construct a Lagrange's function F(x, y, z) = f(x, y, z) + λ g(x, y, z) + α h(x, y, z), where λ and α are non-zero parameter to be determined, known as Lagrange's multipliers or undetermined multipliers.

3. For stationary points of F we have dF=0. Therefore the stationary points of F are given by

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x}+\alpha\frac{\partial h}{\partial x}=0,\ \frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y}+\alpha\frac{\partial h}{\partial y}=0 \text{ and } \frac{\partial f}{\partial z}+\lambda\frac{\partial g}{\partial z}+\alpha\frac{\partial h}{\partial z}=0 \ \ldots\ldots (4)$$

4. Eliminating x, y, λ and α from (2), (3) and (4), we get an equation in u, the roots of which give the stationary values of u=f(x, y, z).

Generally we determine the values of x, y and z in terms of λ and α from equation (4) and put them in the given conditions (2) and (3), which give the values of λ and α. Again putting the values of λ and α in x, y and z obtained from (4) give the stationary points. These points when substituted in the given function give the extremum values.

Note: If only one condition g(x, y, z)=0 is given the Lagrange's function is written as

F(x, y, z) = f(x, y, z) + λ g(x, y, z)

and proceed.

Drawbacks of Lagrange's Method:

(i) The above method does not indicate the nature of the stationary values obtained (i. e. whether they are maximum or minimum), i. e. the nature of the stationary points cannot be determined.

(ii) In general, the nature of the stationary points is determined from the physical or the geometrical considerations of the problem. The nature of these points may also be determined by considerations of the second derivatives, when x, y, z are connected by one relation only.

(iii) The above method may be used even when only one condition is given. It may also be used for stationary values of a function of two variables f(x, y) under one condition g(x, y)=0.

Examples:

1. Use Lagrange's method of multipliers to find the smallest and largest value of x+2y on the circle $x^2+y^2=1$. [IPU 2011]

2. Find the volume of the greatest rectangular parallelopiped that can be inscribed inside the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. [IPU 2006]

3. Find a point upon the plane ax+by+cz=p at which the function $f=x^2+y^2+z^2$ has a minimum value and find the minimum f. [IPU 2004, 2006]

4. The temperature T at any point (x, y, z) of space is given by $T=400xyz^2$, find the highest temperature at the surface of the sphere $x^2+y^2+z^2=1$. [IPU 2005]

5. Find the maximum and minimum distances of the point (3, 4, 12) from the sphere $x^2+y^2+z^2=1$. [IPU 2011]

6. Divide 120 into three parts in such a way that the sum of their products taken two at a time is maximum.

7. Find the minimum value of x^2+y^2 subject to the condition ax+by=c.

8. Find the minimum value of $f(x, y, z)=x^2+y^2+z^2$ subject to the condition x+y+z=1.

9. Determine the values of x, y, z that maximize the function $3x+5y+z-x^2-y^2-z^2$ subject to the constraint x+y+z=6.

10. Find the minimum value of $x^2 + y^2 + z^2$, given that ax +by +cz = p.

11. Find the maxima and minima of the function u = $x^2 + y^2 + z^2$ subject to conditions $ax^2 + by^2 + cz^2$=1 and lx +my +nz =0.

12. In a plane triangle ABC find the maximum value of cosAcosBcosC.