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Lecture Notes

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Part-2

B. Sc. Maths (H)

Paper-III

BETA AND GAMMA FUNCTIONS

Distribution of Lectures

1. Gamma Function.
2. Fundamental Properties of Gamma Function.
3. Another Representation of Gamma Function.
4. Extension of Gamma Function for Negative Non-integers.
5. Examples on Gamma Function.
6. Beta Function.
7. Properties of Beta Function.
8. Another Representation of Beta Function.
9. Relation Between Beta and Gamma Functions.
10. To Evaluate $\int_0^{\frac{\pi}{2}} \sin^p(x).\cos^q(x)dx$.
11. Duplication Formula.
12. Properties of Beta Function.

GAMMA FUNCTION

Gamma Function: The gamma function Γx (read as gamma x) is defined by the improper integral $\Gamma x = \int_0^\infty t^{x-1} e^{-t} dt; x > 0$.

The integral converges when x>0. It is also known as **Eulerian Integral of the Second Kind** or the **Generalised Factorial Function**.

Corollary:

1. **Fundamental Property of Gamma Function**: $\Gamma(x+1) = x\Gamma x$

2. $\Gamma(x+1) = x(x-1)(x-2)\ldots\ldots\ldots$

3. When x is a +ve integer, $\Gamma(x+1) = x(x-1)(x-2)\ldots\ldots\ldots 1.\Gamma 1 =$ x! $\Gamma 1 \Rightarrow \Gamma(x+1) = x!$.

4. $\Gamma 1 = 1$

5. $\Gamma n \Gamma(1-n) = \dfrac{\pi}{\sin(n\pi)}$

Another Representation of Gamma Function: $\Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1} dx; n > 0$

Standard Form of Gamma Function: $\Gamma n = \int_0^1 \left(\log \dfrac{1}{x} \right)^{n-1} dx; n > 0$

Extension of Definition of Gamma Function for x<0:

$\because \Gamma(x+1) = x\Gamma x$

$$\therefore \Gamma x = \frac{\Gamma(x+1)}{x}; (x \neq 0)$$

$$= \frac{(x+1)\Gamma(x+1)}{x(x+1)} = \frac{\Gamma(x+2)}{x(x+1)}; (x \neq 0, -1)$$

$$= \frac{(x+2)\Gamma(x+2)}{x(x+1)(x+2)} = \frac{\Gamma(x+3)}{x(x+1)(x+2)}; (x \neq 0. -1, -2)$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \frac{\Gamma(x+k+1)}{x(x+1)(x+2)\ldots\ldots\ldots(x+k)}; (x \neq 0, -1, -2, -3, \ldots\ldots\ldots, -k) -----(1)$$

We use result (1) to define the gamma function for negative x, choosing for k the smallest integer such that (x+k+1)>0.

Example: To find $\Gamma(-2.5)$.

We take k such that (x+k+1)>0, i.e. -2.5+k+1>0 implies k>1.5.

We choose k=2.

$$\therefore \Gamma(-2.5) = \frac{\Gamma(-2.5+2+1)}{(-2.5)(-1.5)(-0.5)} = \frac{\Gamma 0.5}{-(2.5)(1.5)(0.5)} = \frac{\Gamma(0.5)}{-1.875}$$

Now we obtain the value of $\Gamma(0.5)$ from the table of gamma functions. A table of numerical values of Γx for $1 < x \leq 2$ is given in "**Peirce's Tables**."

Note: **1**. $\Gamma 0$ and Γ (-ve integers) have infinite values.

2. $\Gamma(0.5) = \Gamma(1/2) = \sqrt{\pi} = 1.772$.

3. $\Gamma(-0.5) = \Gamma(-1/2) = -2\sqrt{\pi} = -3.545$.

Important Formula: If n is a +ve integer, then

$$\Gamma(n+\frac{1}{2})=\frac{1.3.5.7.........(2n-1)}{2^n}\sqrt{\pi}\ .$$

Solved Examples on Gamma Functions

Example: Show that $\int_0^\infty x^4 e^{-x^2}dx=\frac{3}{8}\sqrt{\pi}$.

Proof: Put x^2=t implies 2xdx=dt. Therefore

$$\int_0^\infty x^4 e^{-x^2}dx=\frac{1}{2}\int_0^\infty x^3 e^{-t}dt=\frac{1}{2}\int_0^\infty t^{3/2}e^{-t}dt=\frac{1}{2}\int_0^\infty t^{\frac{5}{2}-1}e^{-t}dt=\frac{1}{2}\Gamma(5/2)$$

$$=\frac{1}{2}\Gamma(3/2+1)=\frac{1}{2}.\frac{3}{2}.\Gamma(3/2)=\frac{3}{4}\Gamma(1+\frac{1}{2})=\frac{3}{8}\sqrt{\pi}.$$

Example: Show that $\int_0^\infty 4x^4 e^{-x^4}dx=\Gamma(5/4)$.

Proof: Put x^4=t implies $4x^3$dx=dt. Therefore,

$$\int_0^\infty 4x^4 e^{-x^4}dx=\int_0^\infty xe^{-t}dt=\int_0^\infty t^{1/4}e^{-t}dt=\int_0^\infty t^{\frac{5}{4}-1}e^{-t}dt=\Gamma(5/4).$$

Example: Show that $\int_0^\infty e^{-k^2x^2}dx=\frac{\sqrt{\pi}}{k}$.

Proof: Put k^2x^2=t implies 2xdx=dt/k^2. Therefore,

$$\int_0^\infty e^{-k^2x^2}dx=\int_0^\infty x^{-1}e^{-t}\frac{dt}{k}=\frac{1}{k^2}\int_0^\infty\left(\frac{t}{k^2}\right)^{-1/2}e^{-t}dt=\frac{1}{k}\int_0^\infty t^{\frac{1}{2}-1}e^{-t}dt=\frac{1}{k}\Gamma\left(1/2\right)=\frac{\sqrt{\pi}}{k}.$$

Example: Prove that $\int_0^\infty e^{-x^2} x^{2n-1} dx = \frac{\Gamma n}{2}; n > 0.$

Proof: Put

$$x^2 = t \Rightarrow 2xdx = dt \Rightarrow xdx = \frac{dt}{2} \qquad \therefore \int_0^\infty e^{-x^2} x^{2n-1} dx = \int_0^\infty e^{-x^2} x^{2n-2} xdx = \frac{1}{2}\int_0^\infty e^{-t} t^{n-1} dt = \frac{\Gamma n}{2}.$$

Example: Prove that $\int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx = \Gamma n; n > 0.$

Proof: Put

$$\log \frac{1}{x} = t \Rightarrow \frac{-1}{x} dx = dt \Rightarrow dx = -e^{-t} dt$$

Therefore,

$$\int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx = \int_\infty^0 (t)^{n-1} (-e^{-t}) dt = \int_0^\infty t^{n-1} e^{-t} dt = \Gamma n.$$

Example: If n is a +ve integer, then $\Gamma(n + \frac{1}{2}) = \frac{1.3.5.7.........(2n-1)}{2^n}\sqrt{\pi}$.

Proof: We have $\Gamma\left(n + \frac{1}{2}\right) = \Gamma\left(\frac{2n+1}{2}\right) = \Gamma\left(\frac{2n+1}{2} - 1 + 1\right) = \Gamma\left(\frac{2n-1}{2} + 1\right)$

$$= \left(\frac{2n-1}{2}\right)\Gamma\left(\frac{2n-1}{2}\right) = \left(\frac{2n-1}{2}\right)\Gamma\left(\frac{2n-3}{2} + 1\right) = \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\Gamma\left(\frac{2n-3}{2}\right)$$

$$= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)........\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)........\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi} = \frac{1.3.5...........(2n-3).(2n-1)}{2^n}\sqrt{\pi}$$ Proved.

Example: Prove that $\int_0^\infty \frac{x^c}{c^x} dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}; c > 0.$

Proof: Since $c^x = e^{\log c^x} = e^{x \log c}$. Put $x \log c = t \Rightarrow dx = \frac{dt}{\log c}$ and $x = \frac{t}{\log c}$.

$$\therefore \int_0^\infty \frac{x^c}{c^x} dx = \int_0^\infty \frac{t^c}{(\log c)^c} . e^{-t} \frac{dt}{(\log c)} = \frac{1}{(\log c)^{c+1}} \int_0^\infty e^{-t} t^{c+1-1} dt = \frac{\Gamma(c+1)}{(\log c)^{c+1}}.$$

Example: Prove that $\int_0^\infty x^{y-1} e^{-x} (\log x)^n dx = \frac{d^n (\Gamma n)}{dy^n}$.

Proof: Since $\Gamma y = \int_0^\infty x^{y-1} e^{-x} dx = \int_0^\infty \frac{e^{-x}}{x} x^y dx \quad \Rightarrow \Gamma y = \int_0^\infty \frac{e^{-x}}{x} x^y dx$

$$\Rightarrow \frac{d\Gamma y}{dy} = \int_0^\infty \frac{e^{-x}}{x} x^y (\log x) dx$$

$$\Rightarrow \frac{d^2 \Gamma y}{dy^2} = \int_0^\infty \frac{e^{-x}}{x} x^y (\log x)^2 dx$$

$$\Rightarrow \frac{d^3 \Gamma y}{dy^3} = \int_0^\infty \frac{e^{-x}}{x} x^y (\log x)^3 dx$$

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$$\Rightarrow \frac{d^n \Gamma y}{dy^n} = \int_0^\infty \frac{e^{-x}}{x} x^y (\log x)^n dx = \int_0^\infty x^{y-1} e^{-x} (\log x)^n dx.$$

Example: Prove that $\int_0^1 y^{q-1} \left(\log \frac{1}{y} \right)^{p-1} dy = \frac{\Gamma p}{q^p}; p, q > 0.$

Proof: Put $\log \frac{1}{y} = t \Rightarrow y = e^{-t} \Rightarrow dy = -e^{-t} dt$.

Therefore, $\int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1} dy = \int_\infty^0 -\left(e^{-t}\right)^q t^{p-1}dt = \int_0^\infty \left(e^{-t}\right)^q t^{p-1}dt$

Again put $tq = z \Rightarrow dt = \frac{dz}{q}$. $\therefore \int_0^\infty \left(e^{-t}\right)^q t^{p-1}dt = \int_0^\infty \left(e^{-z}\right)\left(\frac{z}{q}\right)^{p-1}\frac{dz}{q} = \frac{1}{q^p}\int_0^\infty e^{-z}z^{p-1}dz = \frac{\Gamma p}{q^p}$.

Example: Prove that $\int_0^1 x^m(\log x)^n dx = \frac{(-1)^n.n!}{(m-1)^{n+1}}$; where n is a positive integer and m>-1.

Proof: Put $\log x = z \Rightarrow x = e^z \Rightarrow dx = e^z dz$.

Therefore, $\int_0^1 x^m(\log x)^n dx = \int_\infty^0 e^{zm}.z^n\frac{dz}{e^z} = -\int_0^\infty e^{z(m-1)}z^n dz = -\int_0^\infty e^{-(1-m)z}z^n dz$

Again put $(1-m)z = t \Rightarrow (1-m)dz = dt, z = \frac{t}{(1-m)}$.

$$\therefore -\int_0^\infty e^{-(1-m)z}z^n dz = -\int_0^\infty e^{-t}\left(\frac{t}{1-m}\right)^n\frac{dt}{(1-m)} = -\int_0^\infty \frac{e^{-t}t^n}{(1-m)^{n+1}}dt$$

$$= \frac{-1}{(1-m)^{n+1}}\int_0^\infty e^{-t}t^{n+1-1}dt = \frac{-\Gamma(n+1)}{(-1)^{n+1}(m-1)^{n+1}} = (-1)^{n+2}\frac{\Gamma(n+1)}{(m-1)^{n+1}} = \frac{(-1)^n.n!}{(m-1)^{n+1}}.$$

Example: Prove that $\int_0^\infty x^{2n-1}e^{-ax^2}dx = \frac{\Gamma n}{2a^n}$.

Proof: Put $ax^2 = t \Rightarrow xdx = \frac{dt}{2a}$. Therefore, $\int_0^\infty x^{2n-1}e^{-ax^2}dx = \int_0^\infty x^{2n-2}e^{-ax^2}xdx = \int_0^\infty (x^2)^{n-1}e^{-ax^2}xdx$

$$= \int_0^\infty \left(\frac{t}{a}\right)^{n-1}e^{-t}\frac{dt}{2a} = \int_0^\infty \frac{t^{n-1}e^{-t}}{2a^n}dt = \frac{1}{2a^n}\Gamma n.$$

BETA FUNCTION

Beta Function: The beta function $\beta(m,n)$ is defined by $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx; m, n > 0$.

The integral converges when m and n are +ve numbers, integral or fractional. Beta function is also known as the **Eulerian Integral of First Kind**.

Note: $\beta(m,n) = \beta(n,m)$.

Another Representations of Beta Function:

1. $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$ [Put $x = \frac{t}{1+t}$.]

2. $\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$ [Put $x = \tan^2\theta$.]

3. $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta.\cos^{2n-1}\theta d\theta$ [Put $x = \sin^2\theta$.]

Relation Between Beta and Gamma Functions: $\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$.

To Evaluate: $\int_0^{\frac{\pi}{2}} \sin^p x.\cos^q x dx$

$$\int_0^{\frac{\pi}{2}} \sin^p x.\cos^q x dx = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)} = \frac{\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)}{2}$$

Duplication Formula: $\Gamma n\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}.\Gamma(2n)}{2^{2n-1}}$; n is +ve integers.

Relations Between Different Beta Functions:

1. $\beta(m+1,n) = \frac{m}{m+n}\beta(m,n)$

2. $\beta(m,n+1) = \frac{n}{m+n}\beta(m,n)$

3. $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

4. $n\beta(m+1,n) = m\beta(m,n+1)$

Objective:

1. $\int_0^{\infty} \frac{x^{n-1}}{(1+x)} dx = \frac{\pi}{\sin(n\pi)}$

2. $\int_0^{\frac{\pi}{2}} \tan^n x dx = \frac{\pi}{2}\sec\frac{n\pi}{2}; |n| < 1.$

3. $\Gamma n\Gamma(1-n) = \frac{\pi}{\sin(n\pi)}$

4. $\int_0^1 x^m (1-x^n)^p dx = \frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right)$

5. $\iint_D x^{m-1} y^{n-1} dxdy = \frac{\Gamma m\Gamma n}{\Gamma(m+n+1)} a^{m+n}$, where D is the domain $x \geq 0, y \geq 0$ and $x + y \leq a$.

6. **Dirichlet Integral**: $\iiint_V x^{l-1}y^{m-1}z^{n-1}dxdydz = \frac{\Gamma l \Gamma m \Gamma n}{\Gamma(l+m+n+1)}$ where V is the region $x \geq 0$, $y \geq 0$, $z \geq 0$ and $x+y+z \leq 1$.

7. $\int_0^\infty x^{n-1}\cos(ax)dx = \frac{\Gamma n}{a^n}\cos\frac{n\pi}{2}$

8. $\int_0^\infty x^{n-1}\sin(ax)dx = \frac{\Gamma n}{a^n}\sin\frac{n\pi}{2}$

9. $\int_a^b (x-a)^m (b-x)^n\, dx = (b-a)^{m+n+1}\beta(m+1,n+1)$

10. $\int_0^{\frac{\pi}{2}}\sqrt{\tan x}dx = \int_0^{\frac{\pi}{2}}\sqrt{\cot x}dx = \frac{\pi}{\sqrt{2}}$

Solved Examples on Beta Function

Example: Proved that $\beta(n,n) = \frac{\sqrt{\pi}\Gamma n}{2^{2n-1}\Gamma\left(n+\frac{1}{2}\right)}$.

Proof: $R.H.S. = \frac{\sqrt{\pi}\Gamma n}{2^{2n-1}\Gamma\left(n+\frac{1}{2}\right)} = \frac{\sqrt{\pi}\Gamma n}{2^{2n-1}\frac{1.3.5..........(2n-1)}{2^n}\sqrt{\pi}}$

$= \frac{\Gamma n}{2^{n-1}1.3.5..........(2n-1)} = \frac{\Gamma n\{2.4.6.........(2n+2)\}(2n+1)}{2^{n-1}\{1.2.3.4.5..........(2n-1).2n.(2n+1).2n.(2n+2)\}}$

$= \frac{\Gamma n 2^{n+1}(n+1)!(2n+1)}{2^{n-1}(2n+2)!} = \frac{4.\Gamma n.(n+1).n!(2n+1)}{(2n+2).(2n+1).2n.(2n-1)!}$

$$=\frac{\Gamma n.n!}{n.(2n-1)!}=\frac{\Gamma n\Gamma(n+1)}{n\Gamma(2n)}=\frac{\Gamma n.n.\Gamma n}{n.\Gamma(2n)}=\frac{\Gamma n\Gamma n}{\Gamma(2n)}=\beta(n,n).$$

Example: Prove that $\beta(m,\frac{1}{2})=2^{2m-1}\beta(m,m)$.

Proof: $\beta(m,\frac{1}{2})=\dfrac{\Gamma m\Gamma\frac{1}{2}}{\Gamma\left(m+\frac{1}{2}\right)}=\dfrac{\Gamma m\Gamma\frac{1}{2}}{\frac{1.3.5.........(2m-1)}{2^m}\Gamma\frac{1}{2}}$

$$=\frac{2^m\Gamma m}{1.3.5.........(2m-1)}=\frac{2^m\{2.4.6.........2m\}\Gamma m}{1.2.3.4.5.6........(2m-1).(2m)}$$

$$=\frac{2^m 2^m m!\Gamma m}{(2m)!}=\frac{2^{2m}\Gamma m\Gamma(m+1)}{(2m)(2m-1)!}=\frac{2^{2m-1}\Gamma m.m.\Gamma m}{m\Gamma(2m)}=2^{2m-1}\beta(m,m).$$

Example: Prove that $\displaystyle\int_0^1\frac{dx}{\sqrt{1-x^n}}=\frac{\sqrt{\pi}}{n}\frac{\Gamma\frac{1}{n}}{\Gamma(\frac{1}{n}+\frac{1}{2})}$.

Proof: Put $x^{\frac{n}{2}}=\sin\theta\Rightarrow dx=\dfrac{2\cos\theta(\sin\theta)^{2/n}d\theta}{n\sin\theta}$

Therefore, $\displaystyle\int_0^1\frac{dx}{\sqrt{1-x^n}}=\int_0^{\pi/2}\frac{2\cos\theta(\sin\theta)^{2/n}d\theta}{n\sin\theta\cos\theta}=\frac{2}{n}\int_0^{\pi/2}\sin^{\frac{2}{n}-1}\theta d\theta$

$$=\frac{2}{n}\int_0^{\pi/2}\sin^{\frac{2}{n}-1}\theta\cos^0\theta d\theta=\frac{2}{n}\frac{\Gamma\left(\frac{\frac{2}{n}-1+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{\frac{2}{n}-1+0+2}{2}\right)}=\frac{\sqrt{\pi}}{n}\frac{\Gamma\frac{1}{n}}{\Gamma(\frac{1}{n}+\frac{1}{2})}.$$

Example: Prove that $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$.

Proof: Same as above for n=4.

Example: Prove that $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}}{4} \frac{\Gamma\frac{3}{4}}{\Gamma(\frac{5}{4})}$.

Proof: Put $x^2 = \sin\theta \Rightarrow dx = \frac{\cos\theta d\theta}{2\sqrt{\sin\theta}}$

Therefore, $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} = \int_0^{\pi/2} \frac{\sin\theta \cos\theta d\theta}{2\sqrt{\sin\theta}\cos\theta} = \frac{1}{2}\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}+0+2}{2}\right)} = \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}.$$

Example: Prove that $\int_0^1 \frac{x^{2a} dx}{\sqrt{1-x^2}} = \frac{\sqrt{\pi}}{2} \frac{\Gamma(a+\frac{1}{2})}{\Gamma a}$.

Proof: Put $x = \sin\theta$.

Example: Prove that $\int_0^\infty \frac{dy}{1+y^4} = \frac{\pi}{2\sqrt{2}}$.

Proof: Put $y^2 = \tan\theta \Rightarrow dy = \frac{\sec^2\theta d\theta}{2\sqrt{\tan\theta}}$

Therefore, $\int_0^\infty \frac{dy}{1+y^4} = \int_0^{\pi/2} \frac{\sec^2\theta d\theta}{2\sqrt{\tan\theta}\sec^2\theta} = \frac{1}{2}\int_0^{\pi/2} (\sin\theta)^{-1/2}(\cos\theta)^{1/2} d\theta$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{\frac{-1}{2}+1}{2}\right)\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{\frac{-1}{2}+\frac{1}{2}+2}{2}\right)} = \frac{1}{4}\frac{\Gamma\frac{1}{4}\Gamma\frac{3}{4}}{\Gamma 1} = \frac{1}{4}\frac{\sqrt{\pi}\Gamma\left(2.\frac{1}{4}\right)}{2^{\frac{2}{4}-1}} = \frac{1}{4}\frac{\sqrt{\pi}\sqrt{\pi}}{2^{\frac{1}{2}-1}} = \frac{\sqrt{2}}{4}\pi = \frac{\pi}{2\sqrt{2}}.$$

Example: Prove that $\int_0^\pi \sqrt{\tan\theta}d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\sqrt{2}}.$

Proof: Do Yourself