Dr. Dharmendra Kumar Yadav

Assistant Professor of Mathematics

M. L. S. College, Sarisab-Pahi, Madhubani

On Deputation @ R. K. College, Madhubani

Lecture Notes

23/03/2020 to 25/03/2020 [M.L.S. College]

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Part-I

B. Sc. Maths(H)

Paper-I

De Moivre's Theorem

De Moivre’s Theorem

Introduction: Abraham De Moivre (1667-1754) was born in France, but fled to England in 1688, after being imprisoned for his religious beliefs. A brilliant mathematician, he was unable to gain a university appointment (because he was born in France) and escape his life of poverty, gaining only a meagre income as a private tutor. He was friends with Sir Isaac Newton and Edmund Halley (1656 - 1742), was elected to the Royal Society in England, and to the Academies of Paris and Berlin, yet in spite of the support of the great Leibniz (1646 - 1716), urged by Jacques Bernoulli, he never gained a university appointment and died in poverty. In spite of this, he made many discoveries in mathematics, some of which are attributed to others (For instance, Stirling's Formula for factorial approximations was known earlier by De Moivre).

Imaginary Unit ‘i’: After the introduction of imaginary unit ‘i’ with property $\sqrt{-1} = i$, by Swiss mathematician Leonhard Euler(1707-1783) in 1748 A.D., a new number system called the “Complex Numbers System” came into existence. Following this concept De Moivre gave De Moivre’s formula and German mathematician Carl Friedrich Gauss(1777-1855) proved “the Fundamental theorem of Algebra” in 1799A.D., which was given by the Dutch mathematician Albert Girard(1595-1632) possibly in 1625 A.D. Though Euler gave ‘i’ in 1748, the existence of complex numbers was not completely accepted until the geometrical interpretation had been described by Caspar Wessel in 1799. It was rediscovered several years later and popularized by Carl Friedrich Gauss. Gauss proved “the Fundamental Theorem of Algebra” in 1799, which was given by Albert Girard in about 1625. Girard accepted square root of negative numbers and said that this will enable us to get as many roots as the degree of the polynomial equation.

Indian mathematician Mahavira first stated this difficulty in his book ‘Ganitasar Sangraha’ in 850 A.D. He stated that ‘a negative (quantity) is not a square (quantity), therefore no square root.’ Another Indian mathematician Bhaskara also writes in his book ‘Bijaganita’ in 1150 A. D. that ‘there is no square root of a negative quantity, for it is not a square.’

De Moivre's Theorem:

For any x (complex number or real number),

(i) If n is any integer (positive, negative or zero), then $(\cos x + i \sin x)^n = \cos(nx) + i \sin(nx)$, and

(ii) If n is a fraction / rational number (positive or negative), then one of the values of

$$(\cos x + i \sin x)^n \text{ is } \cos(nx) + i \sin(nx).$$

In other words, whatever be the value of n positive or negative, integral or fractional,

$$\cos(nx) + i \sin(nx) \text{ is one of the values of } (\cos x + i \sin x)^n.$$

Proof: Although historically proven earlier, De Moivre's formula can easily be derived from Euler's formula

$$e^{ix} = \cos x + i \sin x$$

by applying the exponential law

$$(e^{ix})^n = e^{ixn}.$$

Then, by Euler's formula,

$$e^{i(nx)} = \cos(nx) + i \sin(nx)$$

Proof by induction:

Case-I: When n is an integer, to prove it we consider three sub-cases.

(i) For $n > 0$, we proceed by principle of mathematical induction. When $n = 1$, the result is clearly true. For our hypothesis, we assume that the result is true for some positive integer k. That is, we assume

$$(\cos x + i\sin x)^k = \cos(kx) + i\sin(kx)$$

Now, consider the case for $n = k + 1$. For this we have,

$$(\cos x + i\sin x)^{k+1} = (\cos x + i\sin x)^k(\cos x + i\sin x)$$

$$= [\cos(kx) + i\sin(kx)].[(\cos x + i\sin x)]$$

$$= [\cos(kx).\cos x - \sin(kx).\sin x] + i[\cos(kx).\sin x + \sin(kx).\sin x]$$

$$= \cos(kx + x) + i\sin(kx + x)$$

$$= \cos(k+1)x + i\sin(k+1)x$$

We find that the result is true for $n = k + 1$ when it is true for $n = k$. Therefore by the principle of mathematical induction the result is true for all positive integers $n \geq 1$.

(ii) When $n = 0$ the formula is true since $\cos(0x) + i\sin(0x) = 1 + i0 = 1$, and $z^0 = 1$.

(iii) When $n < 0$, let us consider a positive integer m such that $n = -m$. So

$$(\cos x + i\sin x)^n = (\cos x + i\sin x)^{-m}$$

$$= \frac{1}{(\cos x + i\sin x)^m}$$

$$= \frac{1}{(\cos mx + i\sin mx)}$$

$$= \frac{1}{(\cos mx + i\sin mx)} . \frac{(\cos mx - i\sin mx)}{(\cos mx - i\sin mx)}$$

$$= \frac{(\cos mx - i\sin mx)}{(\cos^2 mx + \sin^2 mx)}$$

$$= \cos(-mx) + i\sin(-mx)$$

$$= \cos(nx) + i\sin(nx)$$

i.e., the result is also true for negative integers. This completes the proof of part (i).

Case-II: When n is a fraction or a rational number. Let n = p/q in its lowest form, where p is an integer positive or negative and q is a positive integer. Then,

$$(\cos p\frac{\theta}{q} + i\sin p\frac{\theta}{q})^q = \cos p\theta + i\sin p\theta \quad \text{[by part (i)]}$$

Therefore taking the qth root of both sides, we get

$(\cos p\frac{\theta}{q} + i\sin p\frac{\theta}{q})$ is one of the qth root of ($\cos p\theta + i\sin p\theta$).

But $(\cos p\theta + i\sin p\theta) = (\cos\theta + i\sin\theta)^p$. Hence $(\cos p\frac{\theta}{q} + i\sin p\frac{\theta}{q})$ is one of the qth roots of

$(\cos\theta + i\sin\theta)^p$

i.e., $(\cos p\frac{\theta}{q} + i\sin p\frac{\theta}{q})$ is one of the values of $(\cos\theta + i\sin\theta)^{p/q}$.

Therefore one of the values of $(\cos\theta + i\sin\theta)^n$ is $(\cos n\theta + i\sin n\theta)$.

This completes the proof of part (ii).

Observations:

(1) In caseI, in all three subcases in the proof, there is one and only one value of $(\cos\theta + i\sin\theta)^n$ and that is $(\cos n\theta + i\sin n\theta)$. But in caseII, there are q values of $(\cos\theta + i\sin\theta)^n$ one of which is $(\cos n\theta + i\sin n\theta)$.

(2) This theorem is also true for irrational numbers, in this case there are infinite number of values.

(3) It is also possible to define the power of a complex number for a complex exponent. It too has an infinity of values, but the corresponding points do not , in general, accumulate. They are spread out.

The proof for (2) and (3) is beyond the scope of this book.

Generalization: The formula is actually true in a more general setting than stated above: if z and w are complex numbers, then

$$(\cos z + i\sin z)^w$$

is a multi-valued function while

$$\cos(wz) + i\sin(wz)$$

is not.

This is one of the reason that we state it as $\cos(wz)+i\sin(wz)$ is one value of $(\cos z+i\sin z)^{w}$.

Corollary:

(i) $(\cos\theta+i\sin\theta)^{-n}=\cos n\theta-i\sin n\theta$

(ii) $(\cos\theta-i\sin\theta)^{-n}=\cos n\theta+i\sin n\theta$

(iii) $(\cos\theta-i\sin\theta)^{n}=\cos n\theta-i\sin n\theta$

(iv) $\dfrac{1}{(\cos\theta+i\sin\theta)}=\cos\theta-i\sin\theta$

(v) $\dfrac{1}{(\cos\theta-i\sin\theta)}=\cos\theta+i\sin\theta$.

(vi) $(\cos\theta-i\sin\theta)^{n}=(\cos\theta+i\sin\theta)^{-n}$

(vii) $(\sin\theta+i\cos\theta)=i(\cos\theta-i\sin\theta)$

To put the complex number z=a+ib in De Moivre's form:

Let a=rcosθ…….(1) and b=rsinθ……(2).

Then z = a+ib = r(cosθ+isinθ), where r and θ are given by $r=\sqrt{a^2+b^2}$ and $\theta=\tan^{-1}\dfrac{b}{a}$.

Here r is called the modulus of the complex number and it is taken as positive, θ is called the argument of the complex number. The values of θ which satisfies (1) and (2) and lies between -π and +π is called the principal value of the argument. When we speak of the argument, we mean the principal value unless otherwise stated. If θ is the principal value of the argument, then the general value of the argument is (θ+2nπ), where n is any integer positive or negative.

Example: 1 = cos0+isin0, i=cos(π/2)+isin(π/2), etc.

Note: The formula is important because it connects complex numbers (*i* stands for the imaginary unit) and trigonometry.

To Find the Roots of a Complex Number using De Moivre's Theorem

Theorem (qth root theorem): Show that $(\cos\theta + i\sin\theta)^{1/q}$ has q and only q different values.

Proof: We know that cosθ and sinθ are periodic functions with period 2nπ, where n is an integer.

$\therefore$ $\cos\theta + i\sin\theta = \cos(2n\pi+\theta) + i\sin(2n\pi+\theta)$

$\therefore$ $(\cos\theta + i\sin\theta)^{1/q} = \{\cos(2n\pi+\theta) + i\sin(2n\pi+\theta)\}^{1/q}$

$= \{\cos(\frac{2n\pi+\theta}{q}) + i\sin(\frac{2n\pi+\theta}{q})\}$ ……(i)

Now giving n the values 0, 1, 2, 3, ….,(q-1) successively, we get following q values of $(\cos\theta + i\sin\theta)^{1/q}$:

For n=0, $\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$

For n=1, $\cos\frac{2\pi+\theta}{q} + i\sin\frac{2\pi+\theta}{q}$

For n=2, $\cos\frac{4\pi+\theta}{q} + i\sin\frac{4\pi+\theta}{q}$

For n=3, $\cos\frac{6\pi+\theta}{q} + i\sin\frac{6\pi+\theta}{q}$

……………………………….

……………………………….

For n=q-1, $\cos\frac{2(q-1)\pi+\theta}{q}+i\sin\frac{2(q-1)\pi+\theta}{q}$.

Each of the above value is a root of $(\cos\theta+i\sin\theta)^{1/q}$.

To show that there are only q different values of $(\cos\theta+i\sin\theta)^{1/q}$, let us take n=q+r, where r=0, 1, 2, …., then from (i) we have

$\cos\frac{2n\pi+\theta}{q}+i\sin\frac{2n\pi+\theta}{q} = \cos\frac{2(q+r)\pi+\theta}{q}+i\sin\frac{2(q+r)\pi+\theta}{q}$

$= \cos\frac{2r\pi+\theta}{q}+i\sin\frac{2r\pi+\theta}{q}$; since sin(2π+θ) = sinθ and cos(2π+θ) = cosθ………(ii)

Therefore we see that if we give to n the values greater than (q-1), we donot get new roots and all the values obtained from (ii) are the same as obtained from (i). Hence proof completes.

Note: (1) The roots of $(\cos\theta+i\sin\theta)^{1/q}$ are in G.P. with the first term $\cos\frac{\theta}{q}+i\sin\frac{\theta}{q}$ and common ratio $\cos\frac{2\pi}{q}+i\sin\frac{2\pi}{q}$.

(2) To find the values of $(\cos\theta+i\sin\theta)^{p/q}$, where p/q is in its lowest form, we proceed as follows:

$(\cos\theta+i\sin\theta)^{p/q} = \{\cos(2n\pi+\theta)+i\sin(2n\pi+\theta)\}^{p/q}$

$= \{\cos\frac{p(2n\pi+\theta)}{q}+i\sin\frac{p(2n\pi+\theta)}{q}\}$

Now giving to n in succession 0, 1, 2, 3, …., (q-1), we get the q roots of $(\cos\theta+i\sin\theta)^{p/q}$. Here again there will be q and only q different roots.

(3) No two of the roots either in the main theorem or in note(2) will be equal. They would be equal only when the angles involved are equal or they differ by 2π or a multiple of 2π, which is not the case.

(4) In order to find the distinct values of $(\cos\theta + i\sin\theta)^{p/q}$, always see that p/q is in its lowest form.

WORKING RULE TO FIND THE qth ROOTS OF A COMPLEX NUMBER:

Let the given complex number be z = a+ib, whose qth root is to be found out.

Step-I: Write the given complex number z = a+ib in polar form by putting $a = r\cos\theta$, $b = r\sin\theta$, i,e,

$z = r(\cos\theta + i\sin\theta)$; where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\frac{b}{a}$ (principle value only).

Step-II: Take the qth root and apply the above theorem, we get

$$z^{1/q} = r^{1/q}(\cos\theta + i\sin\theta)^{1/q} = r^{1/q}\{\cos(\frac{2n\pi + \theta}{q}) + i\sin(\frac{2n\pi + \theta}{q})\}$$

where r = 0, 1, 2, ……,(q-1).

Putting r = 0, 1, 2, ….,(q-1), we get the required q roots of the given complex number.

For n=0, it reduces to

$$r^{1/q}\{\cos(\frac{\theta}{q}) + i\sin(\frac{\theta}{q})\}$$

This root is known as the **principal qth root** of z.

To find nth roots of Unity

Since we have $1^{1/n} = (\cos 0° + i\sin 0°)^{1/n} = (\cos 2r\pi + i\sin 2r\pi)^{1/n}$

$= \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}$; where r=0, 1, 2, 3, …., (n-1). [by De moivre's theorem]

Now the nth roots of unity are given for r = 0, 1, 2, ……., (n-1) respectively from the above relation.

Another way to find the Roots of unity

The nth roots of unity are those numbers that satisfy the equation $z^n = 1$.

Since $1 = \cos 2\pi + i \sin 2\pi$, it follows that

$$\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

is an nth root of unity.

But 1 can be written using different arguments as follows:

$1 = \cos 2\pi + i\sin 2\pi$

$= \cos 4\pi + i\sin 4\pi$

$= \cos 6\pi + i\sin 6\pi$

$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$= \cos 2n\pi + i\sin 2n\pi$

Hence dividing the argument in each case by n gives the following nth roots of unity.

$$z = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

$$z = \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}$$

$$z = \cos\frac{6\pi}{n} + i\sin\frac{6\pi}{n}$$

and so on.

Note that arguments increase by $\frac{2\pi}{n}$ each time. The roots of unity are regularly spaced in an Argand diagram.

To find the cube roots of a complex number

Step-I: Write the complex number in polar form $z = r(\cos\theta + i\sin\theta)$.

Step-II: Write z in two more equivalent alternative ways by adding 2π to the argument as

$z = r\{\cos(\theta + 2\pi) + i\sin(\theta + 2\pi)\}$

$z = r\{\cos(\theta + 4\pi) + i\sin(\theta + 4\pi)\}$

Write down the cube roots of z by taking the cube root of r and dividing each of the arguments by 3.

This gives the three cube roots as

$$r^{\frac{1}{3}}\left\{\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right\}$$

$$r^{\frac{1}{3}}\left\{\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)+i\sin\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)\right\}$$

$$r^{\frac{1}{3}}\left\{\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)+i\sin\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)\right\}$$

If $z = r(\cos\theta + i\sin\theta)$ is written in any further alternative ways such as $z = r\{\cos(\theta + 6\pi) + i\sin(\theta + 6\pi)\}$,

this gives a cube root of

$$r^{\frac{1}{3}}\left\{\cos\left(\frac{\theta}{3}+\frac{6\pi}{3}\right)+i\sin\left(\frac{\theta}{3}+\frac{6\pi}{3}\right)\right\}=r^{\frac{1}{3}}\left\{\cos\frac{\theta}{3}+i\sin\frac{\theta}{3}\right\}$$

which is the same as one of the previously mentioned roots. Therefore it is impossible to find any more root.

De Moivre's Formula for Multiple Angles:

Expansion of cos(nx), sin(nx) and tan(nx) in powers of cosx, sinx and tanx, when n is a positive integer

The following is nowadays called De Moivre's formula, clearly known to De Moivre, but never explicitly expressed in this form by him.

$$(\cos n\theta + i\sin n\theta) = (\cos\theta + i\sin\theta)^n$$

Expanding R.H.S. by Binomial theorem, we get

$(\cos n\theta + i\sin n\theta) = \cos^n\theta + {}^nc_1.(\cos\theta)^{n-1}.(i\sin\theta) + {}^nc_2.(\cos\theta)^{n-2}.(i\sin\theta)^2 +$

${}^nc_3.(\cos\theta)^{n-3}.(i\sin\theta)^3+\ldots\ldots+{}^nc_n.(i\sin\theta)^n$

$= \{\cos^n\theta - {}^nc_2.(\cos\theta)^{n-2}.(\sin\theta)^2 + {}^nc_4.(\cos\theta)^{n-4}.(\sin\theta)^4 - \ldots\ldots\}$

$+ \ i\{ \ {}^nc_1.(\cos\theta)^{n-1}.(\sin\theta) - {}^nc_3.(\cos\theta)^{n-3}.(\sin\theta)^3 + {}^nc_5.(\cos\theta)^{n-5}.(\sin\theta)^5 - \ldots\ldots\ldots\}$

Equating real and imaginary parts from both sides, we get

$\cos n\theta = \cos^n\theta - {}^nc_2.\cos^{n-2}\theta.\sin^2\theta + {}^nc_4.\cos^{n-4}\theta.\sin^4\theta - \ldots\ldots\ldots \ (1)$

$\sin n\theta = {}^nc_1.\cos^{n-1}\theta.\sin\theta - {}^nc_3.\cos^{n-3}\theta.\sin^3\theta + {}^nc_5.\cos^{n-5}\theta.\sin^5\theta - \ldots\ldots\ldots(2)$

Replace $\sin^2\theta = 1 - \cos^2\theta$ and $\cos^2\theta = 1 - \sin^2\theta$ in (1) and (2) respectively, we get the expansion of $\cos n\theta$ and $\sin n\theta$ in terms of $\cos\theta$ and $\sin\theta$.

Now we have,

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta} = \frac{{}^nc_1.\cos^{n-1}\theta.\sin\theta - {}^nc_3.\cos^{n-3}\theta.\sin^3\theta + {}^nc_5.\cos^{n-5}\theta.\sin^5\theta - \ldots\ldots}{\cos^n\theta - {}^nc_2.\cos^{n-2}\theta.\sin^2\theta + {}^nc_4.\cos^{n-4}\theta.\sin^4\theta - \ldots\ldots\ldots}$$

Dividing numerator and denominator by $\cos^n\theta$, we get

$$\tan n\theta = \frac{{}^nc_1.\tan\theta - {}^nc_3.\tan^3\theta + {}^nc_5.\tan^5\theta - \ldots\ldots}{1 - {}^nc_2.\tan^2\theta + {}^nc_4.\tan^4\theta - \ldots\ldots\ldots}.$$

To express the tangent of the sum of any number of angles

Expansion of $\tan(\theta_1+\theta_2+\theta_3+\theta_4+\ldots\ldots+\theta_n)$

Example 1

Calculate $\left\{2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)\right\}^5$

Answer:

Using De Moivre's theorem

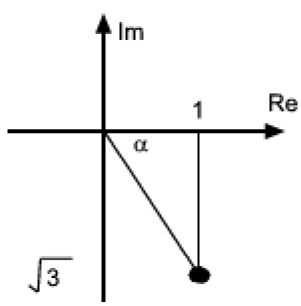
$$\left\{2\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)\right\}^5 = 2^5\left\{\cos\left(5\times\frac{\pi}{5}\right)+i\sin\left(5\times\frac{\pi}{5}\right)\right\}$$
$$= 32\,(\cos\pi + i\sin\pi)$$
$$= -32$$

Example 2

Calculate $(1 - i\sqrt{3})^6$

Answer:

First express $z = 1 - i\sqrt{3}$ in polar form.



$|z| = \sqrt{a^2 + b^2} = \sqrt{1^2 + (-\sqrt{3})^2} =$ 2

As z is in the fourth quadrant arg z = $-\alpha$ where tan $\alpha = \sqrt{3}$, i.e. $\alpha = \frac{\pi}{3}$

So arg z = $\frac{-\pi}{3}$ and z = $2\left\{\cos\left(\frac{-\pi}{3}\right)+i\sin\left(\frac{-\pi}{3}\right)\right\}$

Using De Moivre's theorem

$$z^6 = 2^6\left\{\cos\left(\frac{-6\pi}{3}\right)+i\sin\left(\frac{-6\pi}{3}\right)\right\} = 64\,\{\cos(-2\pi)+i\sin(-2\pi)\} = 64$$

Example 3

Simplify $\dfrac{(1+i)^6}{(1-i\sqrt{3})^4}$

Answer:

First express 1 - i√3 and 1 + i in polar form.

From the previous example 1 - i√3 = $2\left\{\cos\left(\frac{-\pi}{3}\right)+i\sin\left(\frac{-\pi}{3}\right)\right\}$

From an Argand diagram 1 + i = $\sqrt{2}\,(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$

By De Moivre's theorem

$$\begin{aligned}(1-i\sqrt{3})^4 &= 2^4\left\{\cos\left(4\times\frac{-\pi}{3}\right)+i\sin\left(4\times\frac{-\pi}{3}\right)\right\}\\ &= 16\left\{\cos\left(\frac{-4\pi}{3}\right)+i\sin\left(\frac{-4\pi}{3}\right)\right\}\end{aligned}$$

and

$$\begin{aligned}(1+i)^6 &= \sqrt{2}^6\left\{\cos\left(6\times\frac{\pi}{4}\right)+i\sin\left(6\times\frac{\pi}{4}\right)\right\}\\ &= 8\left\{\cos\left(\frac{3\pi}{2}\right)+i\sin\left(\frac{3\pi}{2}\right)\right\}\end{aligned}$$

Hence

$$\begin{aligned}\frac{(1+i)^6}{(1-i\sqrt{3})^4} &= \frac{8}{16}\left\{\cos\left(\frac{3\pi}{2}+\frac{4\pi}{3}\right)+i\sin\left(\frac{3\pi}{2}+\frac{4\pi}{3}\right)\right\}\\ &= \frac{1}{2}\left\{\cos\left(\frac{17\pi}{6}\right)+i\sin\left(\frac{17\pi}{6}\right)\right\}\end{aligned}$$

Example 3: What are each of the five fifth-roots of $\sqrt{3}+i$ expressed in trigonometric form?

$$r = \left|\sqrt{3}+i\right|$$
$$r = \sqrt{\sqrt{3}^2+1^2}$$
$$r = \sqrt{3+1}$$
$$r = 2$$

Since cos $\sqrt{3}/2$ and sin α = ½, α is in the first quadrant and α = 30°. Therefore, since the sine and cosine are periodic,

$$z = r\left(\cos\alpha + i\sin\alpha\right)$$
$$z = 2\left[\cos\left(30^{\circ} + k\cdot 360^{\circ}\right) + i\sin\left(30^{\circ} + k\cdot 360^{\circ}\right)\right]$$

and applying the *n*th root theorem, the five fifth-roots of *z* are given by

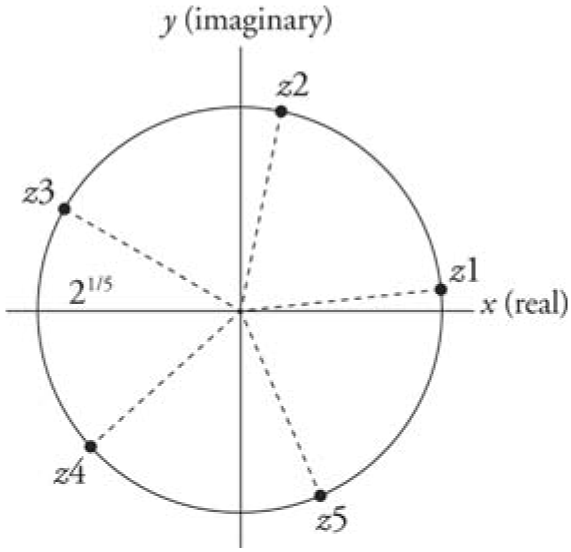
$$2^{1/5}\left[\cos\left(\frac{30^{\circ} + k\cdot 360}{5}\right) + i\sin\left(\frac{30^{\circ} + k\cdot 360}{5}\right)\right]$$

where *k* = 0, 1, 2, 3, and 4

Thus the five fifth-roots are

$$z_1 = 2^{1/5}\left(\cos 6^{\circ} + i\sin 6^{\circ}\right)$$
$$z_2 = 2^{1/5}\left(\cos 78^{\circ} + i\sin 78^{\circ}\right)$$
$$z_3 = 2^{1/5}\left(\cos 150^{\circ} + i\sin 150^{\circ}\right)$$
$$z_4 = 2^{1/5}\left(\cos 222^{\circ} + i\sin 222^{\circ}\right)$$
$$z_5 = 2^{1/5}\left(\cos 294^{\circ} + i\sin 294^{\circ}\right)$$

Observe the even spacing of the five roots around the circle in Figure 1 .



Example 2

Find the cube roots of 1 + i

First express 1 + i in polar form

$|1+i| = \sqrt{2}$ and arg (1 + i) = $\frac{\pi}{4}$

Hence 1 + i can be expressed as $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

But 1 + i can also be expressed as

$$1 + i = \sqrt{2}\left\{\cos\left(\frac{\pi}{4} + 2\pi\right) + i\sin\left(\frac{\pi}{4} + 2\pi\right)\right\} = \sqrt{2}\left\{\cos\left(\frac{9\pi}{4}\right) + i\sin\left(\frac{9\pi}{4}\right)\right\}$$

and

$$1 + i = \sqrt{2}\left\{\cos\left(\frac{\pi}{4} + 4\pi\right) + i\sin\left(\frac{\pi}{4} + 4\pi\right)\right\} = \sqrt{2}\left\{\cos\left(\frac{17\pi}{4}\right) + i\sin\left(\frac{17\pi}{4}\right)\right\}$$

Hence, taking the cube root of the modulus and dividing the argument by 3, the cube roots of 1 + i are

$$z = (2^{1/2})^{1/3}\left(\cos\frac{\pi}{12} + i\ \sin\frac{\pi}{12}\right) = (2^{1/6})\left(\cos\frac{\pi}{12} + i\ \sin\frac{\pi}{12}\right)$$

$$z = (2^{1/6})\left\{\cos\left(\frac{3\pi}{4}\right) + i\ \sin\left(\frac{3\pi}{4}\right)\right\}$$

$$z = (2^{1/6})\left\{\cos\left(\frac{17\pi}{12}\right) + i\ \sin\left(\frac{17\pi}{12}\right)\right\}$$

In this way the nth roots of any complex number can be found.

Example 3

Find the cube roots of z = 64(cos 30° + i sin 30°)

Answer:

This is in polar form. Use 2π = 360° and 4π = 720°

z = 64(cos 30° + i sin 30°)

z can also be written as

z = 64{cos (30 + 360)° + i sin (30 + 360)°}

and

z = 64{cos (30 + 720)° + i sin (30 + 720)°}

Since $64^{1/3} = \sqrt[3]{64} = 4$, the cube roots of z are

4(cos 10° + i sin 10°), 4(cos 130° + i sin 130°), 4(cos 250° + i sin 250°)

Question 3

Find the fourth roots of 81i, that is of $81\left(\cos\frac{\pi}{2} + i\ \sin\frac{\pi}{2}\right)$

Answer

z = $81\left(\cos\frac{\pi}{2}+\text{i}\sin\frac{\pi}{2}\right)$

z = $81\left(\cos\frac{5\pi}{2}+\text{i}\sin\frac{5\pi}{2}\right)$

z = $81\left(\cos\frac{9\pi}{2}+\text{i}\sin\frac{9\pi}{2}\right)$

z = $81\left(\cos\frac{13\pi}{2}+\text{i}\sin\frac{13\pi}{2}\right)$

$r^{1/4}$= 3

The fourth roots of $81\left(\cos\frac{\pi}{2}+\text{i}\sin\frac{\pi}{2}\right)$are

$$3\left(\cos\frac{\pi}{8}+\text{i}\sin\frac{\pi}{8}\right), 3\left(\cos\frac{5\pi}{8}+\text{i}\sin\frac{5\pi}{8}\right),$$
$$3\left(\cos\frac{9\pi}{8}+\text{i}\sin\frac{9\pi}{8}\right) \text{ and } 3\left(\cos\frac{13\pi}{8}+\text{i}\sin\frac{13\pi}{8}\right)$$

Question 4

Find the sixth roots of √3 + i

Answer

The modulus of √3 + i is 2 and the argument is $\frac{\pi}{6}$

The sixth roots are $\sqrt[6]{2}\left(\cos\frac{\pi}{36}+\text{i}\sin\frac{\pi}{36}\right)$, $\sqrt[6]{2}\left\{\cos\left(\frac{13\pi}{36}\right)+\text{i}\sin\left(\frac{13\pi}{36}\right)\right\}$,

$\sqrt[6]{2}\left\{\cos\left(\frac{25\pi}{36}\right)+\text{i}\sin\left(\frac{25\pi}{36}\right)\right\}$, $\sqrt[6]{2}\left\{\cos\left(\frac{37\pi}{36}\right)+\text{i}\sin\left(\frac{37\pi}{36}\right)\right\}$,

$\sqrt[6]{2}\left\{\cos\left(\frac{49\pi}{36}\right)+\text{i}\sin\left(\frac{49\pi}{36}\right)\right\}$and $\sqrt[6]{2}\left\{\cos\left(\frac{61\pi}{36}\right)+\text{i}\sin\left(\frac{61\pi}{36}\right)\right\}$

Example 4

Find the cube roots of unity and plot them on an Argand diagram.

Answer:

Since 1 can be written in polar form as

$1 = \cos 2\pi + i \sin 2\pi$

$1 = \cos 4\pi + i \sin 4\pi$

$1 = \cos 6\pi + i \sin 6\pi$

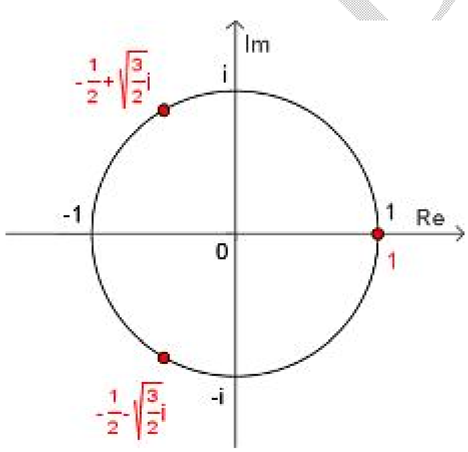
the cube roots of unity are

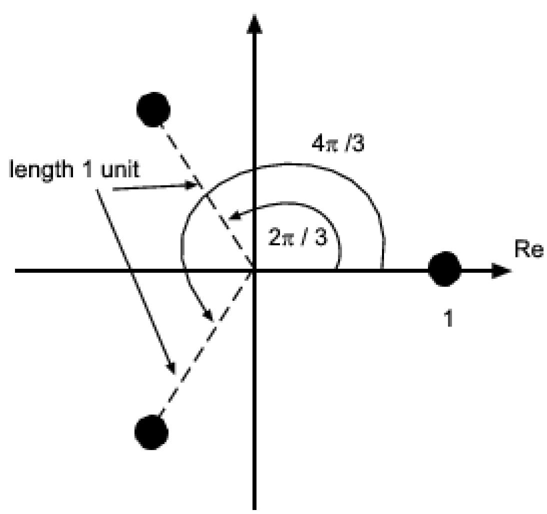
$z = \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} = \frac{1}{2}(-1 + i\sqrt{3})$

$z = \cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3} = \frac{1}{2}(-1 - i\sqrt{3})$

$z = \cos\frac{6\pi}{3} + i \sin\frac{6\pi}{3} = 1$

On the Argand diagram the cube roots of 1.





Question 5

Find the fourth roots of unity and plot them on an Argand diagram.

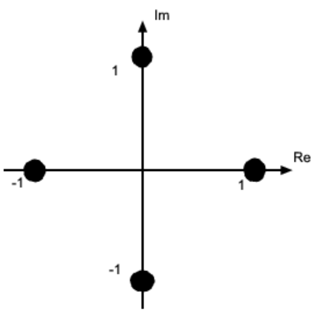
Answer

The solutions are

z = cos 0 + i sin 0 , $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$, cos π+ i sin $\pi$and $\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$

i.e. z = 1, -1, i and -i

On an Argand diagram this gives



Question 6

Find the solutions of the equation $z^6 - 1 = 0$. Plot the answers on an Argand diagram.

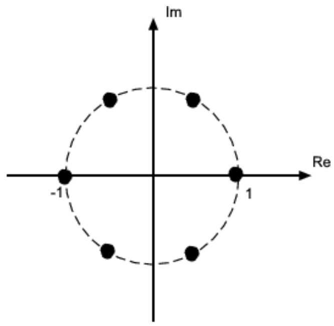
Answer

The solutions are

cos 0 + i sin 0, $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$, $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, cos π+ i sin π,

$\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$ and $\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$

The Argand diagram gives



Worked Out Examples

Example1: Simplify $\dfrac{(\cos\theta + i\sin\theta)^5(\cos 2\theta - i\sin 2\theta)^2}{(\cos 3\theta + i\sin 3\theta)^{11}(\cos 4\theta - i\sin 4\theta)^8}$.

We have by De Moivre's Theorem $(\cos\theta + i\sin\theta)^5 = (\cos 5\theta + i\sin 5\theta)$ and

$(\cos 2\theta - i\sin 2\theta)^2 = (\cos 4\theta - i\sin 4\theta) = \cos(-4\theta) + i\sin(-4\theta)$

$\therefore$ the numerator = $\{\cos 5\theta + i\sin 5\theta\}\{\cos(-4\theta) + i\sin(-4\theta)\}$

= $\cos(5\theta - 4\theta) + i\sin(5\theta - 4\theta) = \cos\theta + i\sin\theta$.

Again we have $(\cos 3\theta + i\sin 3\theta)^{11} = \cos 33\theta + i\sin 33\theta$ and

$(\cos 4\theta - i\sin 4\theta)^8 = \cos 32\theta - i\sin 32\theta = \cos(-32\theta) + i\sin(-32\theta)$.

$\therefore$ the denominator = $(\cos 4\theta - i\sin 4\theta)^8 = \cos 32\theta - i\sin 32\theta = \cos(-32\theta) + i\sin(-32\theta) =$ $\cos\theta + i\sin\theta$.

Therefore the given expression = $\dfrac{\cos\theta + i\sin\theta}{\cos\theta + i\sin\theta}$ =1.